Date: 02 March 2021

VIRTUAL COACHING CLASSES
ORGANISED BY BOS (ACADEMIC), ICAI

FOUNDATION LEVEL
PAPER 3: BUSINESS MATHEMATICS, LOGICAL REASONING & STATISTICS

Faculty: CA Arijit Chakraborty

UNIT OVERVIEW

- **Calculus**
  - Integral Calculus
    - Methods of Substitution
    - Basic Formulas
  - Differential Calculus
    - Basic Laws of Differentiation
    - Some Standard Results
    - Applications

Overview : 8.A.1

- Differentiation is one of the most important fundamental operations in calculus. Its theory primarily depends on the idea of limit and continuity of function.
- To express the **rate of change in any function** we introduce **concept of derivative which involves a very small change in the dependent variable with reference to a very small change in independent s.**
- Thus differentiation is the process of finding the **derivative of a continuous function.** It is defined as the **limiting value of the ratio of the change (increment) in the function corresponding to a small change (increment) in the independent variable (argument) as the later tends to zero.**

8.A.2

- Let y = f(x) be a function.
- If h (or Dx) be the small increment in x and the corresponding increment in y or f(x) be Dy = f(x+h) – f(x) then the derivative of f(x) is defined:
- Lim h --- 0
- F(x+h) – f(x) / h
- = dy/ dx

Limit

The limit is the value that y approaches as x approaches a given value

Differentiation is the process of computing the derivative of a function.

Let f(x) = x2   for x= 3

$$
\begin{aligned}
f'(3) &= \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} \\
&= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \\
&= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \lim_{h \to 0} (6 + h) \\
&= 6
\end{aligned}
$$

Pg 8.6

- **Example:** Differentiate each of the following functions with respect to x:
- A)   3x2   +5x–2
- (a) Let y = f(x) = 3x2 + 5x –2
- = 3 × 2x + 5.1 – 0 = 6x + 5

■ B) Let h(x) = a ^{x} + x ^{a} + a power a

- = a^x log a + ax $^{a-1}$ + 0
- C) Let f(x) =   1/3 x3 -5x2 +6x-2logx+3 .
- =   x2 - 10x + 6 –2 / x

Product rule

Product Rule

If $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$
$$= f(x)g'(x) + g(x)f'(x)$$

or

Let $u = f(x)$ and $v = g(x)$ then

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

F(x) = $(3x^2 — 1)(x2 + 5x+2)$

$$f'(x) = \left(3x^2 - 1\right)\frac{d}{dx}\left(x^2 + 5x + 2\right) + \left(x^2 + 5x + 2\right)\frac{d}{dx}\left(3x^2 - 1\right)$$

$$= \left(3x^2 - 1\right)\left(2x + 5\right) + \left(x^2 + 5x + 2\right)\left(6x\right)$$

$$= 6x^3 + 15x^2 - 2x - 5 + 6x^3 + 30x^2 + 12x$$

$$= 12x^3 + 45x^2 + 10x - 5$$

- D) Let $y = e^x \log x$
- Dy/dx = product rule = $e^x / x + e^x \log x$

- E) $y = 2^x x^5$
- $dy/dx = x^5 2^x \log_e 2 + 5. 2^x x^4$

- F) Let $y = e^x / \log x$ (Division / quotient rule)
- = **(logx) d(e^x) - e^x d(log x)**
- = dx dx
- **(logx)2**

$$\frac{dy}{dx} = \frac{(\log x)\frac{d(e^x)}{dx} - e^x \frac{d(\log x)}{dx}}{(\log x)^2}$$

Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

$$\frac{d}{dx}\left[\frac{Hi}{Ho}\right] = \frac{Ho \cdot dHi \; - \; Hi \cdot dHo}{Ho\,Ho}$$

Calcworkshop.com

quotient of two functions

- Let $h(x) = 2x / 3x^3 + 7$
- Quotient rule
- $= (3x^3 + 7)\,2 \;---\; 2x\,(9x^2) \;/\; (3x^3 + 7)^2$

Examples of differentiations from the 1st principle

- d (c) /dx = 0   where c = constant
- Let $f(x) = x^n$, the $dy/dx= nx^{n-1}$
- $f(x)=e^x$, $dy/dx= e^x$ **Exponential Function**
- The **exponential** constant is an important mathematical constant and is given the symbol **e.** Its **value** is approximately 2.718. It has been found that this **value** occurs so frequently when mathematics is used to model physical and economic phenomena that it is convenient to write simply **e.**
- Let $f(x) = a^x$ $dy/ dx= a^x$ logea
- Let f(x)=   sq root x , then dy/dx= 1 / 2 $x^{½}$
- f(x) =logx , dy/dx= 1/x

8.A.3 SOME STANDARD RESULTS(FORMULAS)- Table: Few functions and their derivatives

e= The number **e** is a mathematical constant approximately equal to 2.71828 and is the base of the natural logarithm,

Function	derivative of the function
$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\log x$	$1/x$
a^x	$a^x \log_e a$
c (a constant)	0

Table: Basic Laws for differentiation

Function	Derivative of the function
(i) $h(x) = c.f(x)$ where c is any real constant (Scalar multiple of a function)	$\frac{d}{dx}\{h(x)\} = c.\frac{d}{dx}\{f(x)\}$
(i) $h(x) = f(x) \pm g(x)$ (Sum/Difference of function)	$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$
(ii) $h(x) = f(x).\ g(x)$ (Product of functions)	$\frac{d}{dx}\{h(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$
(i) $h(x) = \frac{f(x)}{g(x)}$ (Quotient of function)	$\frac{d}{dx}\{h(x)\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
(i) $h(x) = f\{g(x)\}$	$\frac{d}{dx}\{h(x)\} = \frac{df(z)}{dz}.\frac{dz}{dx}$, where $z = g(x)$

Derivatives

- Derivatives measure the pitch of the line that represents a function on a graph, at one particular point on that line. **That means derivatives are the slope.**
- If the independent variable (the "input" variable in a function) is "time," then the derivative is the rate of change, as the velocity.
- If we look at a short section of the line of the function so that the line is nearly straight, **the derivative of that section is the slope of the line.**
- The **slope of a line connecting two points on a function graph approaches the derivative when the interval between the points is zero.**

8.A.4 DERIVATIVE OF A FUNCTION OF FUNCTION

- If y = f [h(x)] then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f'(u) \times h'(x)$
- where u = h(x)

Pg 8.8

- **Example:** Differentiate log (1 + x2) wrt. x
- **Solution:** Let y = log (1 + x 2) = logt when t = 1 + x2
- $2x \; / \; (1+x^2)$

8.A.5 IMPLICIT FUNCTIONS

- A function in the form f(x, y) = 0. For example **$x2y2 + 3xy + y = 0$ where y cannot be directly defined as a function of x is called an implicit function of x.**
- In case of implicit functions if y be a differentiable function of x, no attempt is required to express y as an explicit function of x for finding out dy/ dx
- In such case differentiation of both sides with respect of x and substitution of dy/dx = y1 gives the result. **Thereafter y1 may be obtained by solving the resulting equation.**

Pg 8.9

- **Example:** Find dy/ dx for $x^2y^2 + 3xy + y = 0$
- Differentiating with respect to x we see
- $X^2 \, d/dx \; y^2 + y^2 . 2x + 3 (x. dy/ dx + y) + dy/dx = 0$
- $2y x^2 dy/ dx + 2xy^2 + 3x dy/ dx + 3y + dy/dx = 0$
- $Dy/ dx(2yx^2+3x+1) =$ --- $(2 xy^2 + 3Y)$

8.A.6 PARAMETRIC EQUATION

- When both the variables x and y are expressed in terms of a parameter (a third variable), the involved equations are called parametric equations.

- Dy/ dx = dy/ dt . Dt/ dx
- Example : If $x = at^3$, y = a / t3, find dy/ dx
- dy=dy´ dt=-3a´1 = -1
- dx dt dx t^4 t^6

DERIVATIVE OF A FUNCTION OF FUNCTION

- **Example:** Differentiate log (1 + x2) wrt. x
- **Solution:** Let y = log (1 + x 2) = logt
- when t = 1 + x2
- Dy/ dt = 1/t * dt/dx= 1/ (1+x2) * 2x

8.A.7 LOGARITHMICDIFFERENTIATION

- The procedure is convenient to adopt when the function to be differentiated involves a function in its power or when the function is the product of number of functions
- **Example:** Differentiate x^x w.r.t. x
- **Solution:** let $y = x^x$ Taking logarithm, $\log y = x \log x$
- $= x^x (1 + \log x)$
- This procedure is called logarithmic differentiation.

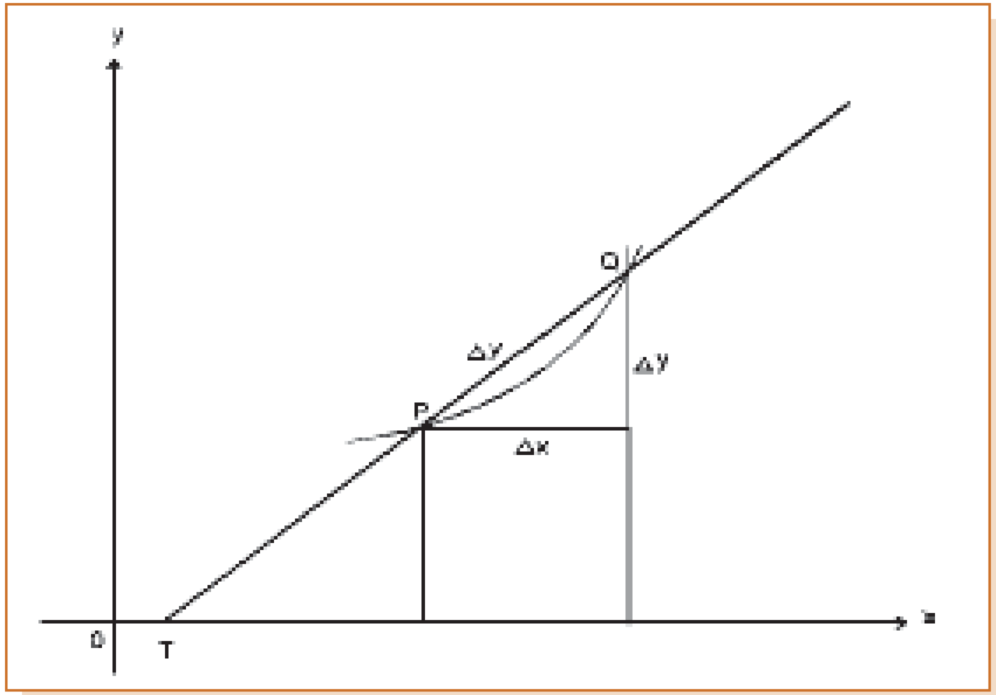
- If $x^m y^n$ = (x+y) power m+n prove that dy/dx = y/x
- Taking log on both sides
-

  log xm yn = (m+n) log (x + y)
- or m log x + n log y = (m+n) log (x+y)
- m/x + x/y dy/dx = (m+n/ x+y) (1+ dy/dx)
- Transposing m/x to RHS and (m+n/x+y) to LHS
- Dy/ dx = y/x

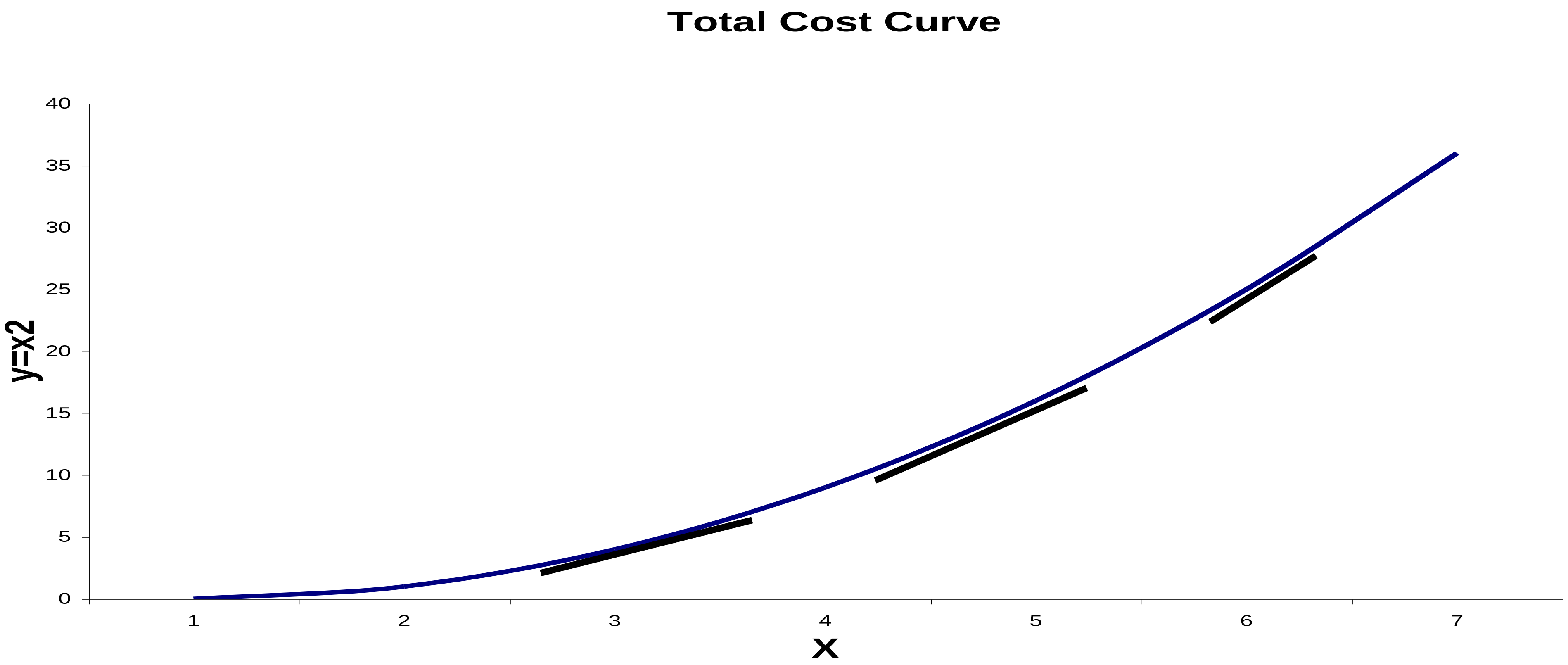
8.A.9 BASIC IDEA ABOUT HIGHER ORDER DIFFERENTIATION

- Let y = f(x) = x 4 + 5x 3 + 2x2 + 9
- dy/ dx = 4x3 + 15 x2 + 4x
- d2y
- Dx2 = 12 x 2 + 30 x
- D3y/ dx3 = 24x + 30

8.A.10 GEOMETRIC INTERPRETATION OF THE DERIVATIVE



The slope of a curve is equal to the slope of the line (or tangent) that touches the curve at that point



which is different for different values of x

- Let f(x) represent the curve in the fig. We take two adjacent pairs P and Q on the curve Let f(x) represent the curve in the fig. We take two adjacent points P and Q on the curve whose coordinates are (x, y) and (x + Dx, y+Dy) respectively. The slope of the chord TPQ is given by dy/dx
- The derivative of f(x) at a point x represents the slope (or sometime called the gradient of the curve) of the tangent to the curve y = f(x) at the point x
- **Example:** Find the gradient of the curve y = 3x2 – 5x + 4 at the point (1, 2).
- Dy/dx = = 6x –5
- At 1,2 ----- 6.1-5= gradient is 1

Applications of Differential Calculus:

- Differentiation helps us to find out the average rate of change in the dependent variable with respect to change in the independent variable.
- It makes differentiation to have applications.
- Various scientific formulae and results involves :
- rate of change in price,
- change in demand with respect change in output,
- change in revenue obtained with respect change in price,
- change in demand with respect change in income, etc.

Pg 8.15

- **Cost Function:** Total cost consists of two parts (i) Variable Cost (ii) Fixed Cost
- If C(X) denotes the cost producing x units of a product then C(x) = V(x) + F(x) , where V(x) denotes the variable cost and F(x) is the fixed cost. Variable cost depends upon the number of units produced (i.e value of x) whereas fixed cost is independent of the level of output x. For example.
- **Average cost (AC or C) = Total cost / output**
- **Average variable cost (AVC)= V.C/ Output**
- **Average Fixed Cost (AFC) = FC/ output**
- **Marginal Cost:** If C(x) the total cost producing x units then the increase in cost in producing one more unit is called marginal cost at an output level of x units and is given as dy/dx
- Marginal Cost (MC) = Rate of change in cost C per unit change in Output at an output level of x units = DC/dx
- To increase profits of a company may decide to increase its production. The question that concerns the management is how will the cost be affected by an increase in

Differentiation in Economics

Application I

- Total Costs = TC = FC + VC
- Total Revenue = TR = P * Q
- π = Profit = TR – TC
- Break even: $\pi = 0$, or TR = TC
- Profit Maximisation: MR = MC

Example 2: pg 8.16= maxima & minima

- The cost function of a company is given by:
- **C(x) = 100 x- 8x2 + x3 / 3**
- where x denotes the output. Find the level of output at which:
- marginal cost is minimum
- average cost is minimum

- M(x) = Marginal Cost = C(x) = d/dx (= 100 x- 8x2 + x3 / 3) = 100 – 16x + x2
- M(x)is maximum or minimum when M¢(x)=-16+2x=0 or,x=8.
- D2x / dx2 = 2 > 0 , so minima at x= 8
- **Avg Cost = Cx / x = 100- 8x + x2/ 3**

- A(x) is maximum or minimum when A¢(x = --8 + 2x/ 3= 0
- X = 12
- D2x/ dx2 = 2/3 > 0
- So AC is minimum at x= 12
- So AC = putting x = 12 in eqn = 52

Revenue Function – pg 8.16

- : Revenue, R(x), gives the total money obtained (Total turnover) by selling x units of a product. If x units are sold at ‘P per unit, then R(x)=P.X
- **Marginal Revenue:** It is the rate of change I revenue per unit change in output. If R is the revenue and x is the output, then MR= DR/ DX
- **Profit function:** Profit P(x), the difference of between total revenue R(x) and total Cost C (x).
- P(X)= R(x) – C(x)
- **Marginal Profit:** It is rate of change in profit per unit change in output
- dP/ dx

Example 3 – pg 8.17

- **Example 3:** A computer software company wishes to start the production of floppy disks. It was observed that the company had to spend ` 2 lakhs for the technical informations. The cost of setting up the machine is ` 88,000 and the cost of producing each unit is ` 30, while each floppy could be sold at ` 45.Find:
- the total cost function for producing x floppies;and
- the break-even point.

Solution

- Given, fixed cost = ` 2,00,000 + ` 88,000 = ` 2,88,000.
  - *If C(x) be the total cost function for producing floppies, then C(x)=30x+2,88,000*
  - *The Revenue function R(x), for sales of x floppies is given by R(x) = 45x. For break-even point, R(x) =C(x)*
- i.e., 45x = 30x + 2,88,000
- i.e., 15x = 2,88,0000 □ x = 19,200, the break-even point

Example 4

- **Example 4:** A company decided to set up a small production plant for manufacturing electronic clocks. The total cost for initial set up (fixed cost) is ` 9 lakhs. The additional cost for producing each clock is ` 300. Each clock is sold at ` 750. During the first month, 1,500 clocks are produced and sold.
- What profit or loss the company incurs during the first month, when all the 1,500 clocks are sold?
- Determine the break-even point.
- (b) Total cost of producing 20 items of a commodity is ` 205, while total cost of producing 10 items is `135. Assuming that the cost function is a linear function, find the cost function and marginal cost function.

Solution:

- The total cost function for manufacturing x Clocks is given by C(x) = Fixed cost + Variable cost to produce x Clocks = 9,00,000 +300x.
- The revenue function from the sale of x clocks in given by R(x) = 750 × x = 750x.
  - *Profit function, P(x)= R(x) –C(x)*
- = 750x – (9,00,000 + 300x) = 450x – 9,00,000
- ☐ Profit, when all 1500 clocks are sold = P(1500) = 450 × 1500 – 9,00,000 = – ` 2,25,000 Thus, there is a loss of ‘ 2,25,000 when only 1500clocks are sold.
  - *At the break-even point, R(x) = C(x) or, 9,00,000 + 300x =750x*
- or, 450x=9,00,000 ☐ x =2,000
- Hence, 2000 clocks have to be sold to achieve the break-even point.

- Let cost function be
- $C(x) = ax+b$, (i)
- x being number of items and a, b being constants.
- Given, $C(x) = 205$ for $x = 20$ and $C(x) = 135$ for $x = 10$. Putting these values in (i),
- $205=20a+b$ (ii)
- $135=10a+b$ (iii)
- (ii) – (iii) gives,
- $70=10a$ or, $a = 7$
- From(iii), $b=135-10a=135-70=65$
- Required cost function is given by $C(x) = 7x+ 65$
- Marginal cost function, $C¢(x) = 7$

- **Marginal Propensity to Consume (MPC):** The consumption function C = F(Y) expresses the **relationship between the total consumption and total Income (Y)**,
- then the **marginal propensity to consume** is defined as the **rate of Change consumption per unit change in Income**
- i.e. dC/ dY . , By consumption we mean expenditure incurred in on Consumption.

- **Marginal Propensity to save (MPS):** Saving,
- S is the difference between income, I and consumption,
- c, i.e ., dS / dY.

Recap

- Limit – concept
- Derivative –concept
- Derivative – addition & subtraction
- Product rule
- Quotient rule
- Implicit functions
- Logarithmic differentiation
- Parametric functions
- Slope –tangent
- Higher order derivatives
- Maxima & minima

Extra content : Minimum and Maximum

- Let's imagine you own a company, and your company's profit can be modeled by the function $P(x) = -10x^2 + 1760x - 50000$, where $P(x)$ is your company's profit, and x is the number of products sold. To find that maximum profit and solve problems similar to this one, we need to be familiar with maximum and minimum points of a function.
- A **maximum point of a function** is the highest point on the graph of a function, or the point that takes on the largest *y*-value. The **minimum point of a function** is the lowest point on the graph of a function, or the point that takes on the smallest *y*-value. Now take a look at the graph below.

THANK YOU